OSCILLATION CRITERIA FOR DELAY EQUATIONS

by

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ABSTRACT

Consider the first-order linear delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$, $\tau(t)$ is non-decreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$,

The most interesting oscillation criteria for Eq.(1), especially in the case where

$$0 < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < 1,$$

are presented.

Key Words: Oscillation; delay differential equations.
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1 Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where the functions $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0, \infty)$), $\tau(t)$ is non-decreasing , $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example, [1-49] and the references cited therein.

By a solution of Eq.(1) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1) is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

In this paper our main purpose is to present the state of the art on the oscillation of all solutions to Eq.(1) especially in the case where

$$0 < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < 1.$$

2 Oscillation Criteria for Eq. (1)

The first systematic study for the oscillation of all solutions to Eq.(1) was made by Myshkis. In 1950 [33] he proved that every solution of Eq.(1) oscillates if

$$\limsup_{t\to\infty} [t - \tau(t)] < \infty \quad \text{and} \quad \liminf_{t\to\infty} [t - \tau(t)] \liminf_{t\to\infty} p(t) > \frac{1}{e}. \tag{C_1}$$

In 1972, Ladas, Lakshmikantham and Papadakis [24] proved that the same conclusion holds if

$$A := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1. \tag{C_2}$$

In 1979, Ladas [23] established integral conditions for the oscillation of Eq.(1) with constant delay. Tomaras [42-44] extended this result to Eq.(1) with variable delay. For related results see Ladde [27-29]. The following most general result is due to Koplatadze and Canturija [18].

If

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{C_3}$$

then all solutions of Eq.(1) oscillate; If

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < \frac{1}{e}, \tag{N_1}$$

then Eq.(1) has a nonoscillatory solution.

In 1982 Ladas, Sficas and Stavroulakis [26] and in 1984 Fukagai and Kusano [10] established oscillation criteria (of the type of conditions $(C_2)$and (C_3)) for Eq. (1) with *oscillating* coefficient $p\,(t)$.

It is obvious that there is a gap between the conditions (C_2) and (C_3) when the limit $\lim\limits_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds$ does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [9] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible nonoscillatory solutions $x(t)$ of Eq.(1). Their result says that all the solutions of Eq.(1) are oscillatory, if $0 < \alpha \leq \frac{1}{e}$ and

$$A > 1 - \frac{\alpha^2}{4}. \tag{C_4}$$

Since then several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$.

In 1991, Jian [16] derived the condition

$$A > 1 - \frac{\alpha^2}{2(1-\alpha)}, \tag{C_5}$$

while in 1992, Yu and Wang [47] and Yu, Wang, Zhang and Qian [48] obtained the condition

$$A > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{C_6}$$

In 1990, Elbert and Stavroulakis [6] and in 1991 Kwong [22], using different techniques, improved (C_4), in the case where $0 < \alpha \leq \frac{1}{e}$, to the conditions

$$A > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2 \tag{C_7}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{C_8}$$

respectively, where λ_1 is the smaller real root of the equation $\lambda = e^{\alpha\lambda}$.

In 1994, Koplatadze and Kvinikadze [19] improved (C_6), while in 1998, Philos and Sficas [34] and in 1999, Zhou and Yu [49] and Jaroš and Stavroulakis [15] derived the conditions

$$A > 1 - \frac{\alpha^2}{2(1-\alpha)} - \frac{\alpha^2}{2}\lambda_1, \tag{C_9}$$

$$A > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2, \tag{C_{10}}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}, \tag{C_{11}}$$

respectively.

Consider Eq.(1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that $p(\tau(t))\tau'(t) \geq \theta p(t)$ eventually for all t. Under this additional condition, in 2000, Kon, Sficas and Stavroulakis [17] and in 2003, Sficas and Stavroulakis [35] established the conditions

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \alpha - \sqrt{(1-\alpha)^2 - 4\Theta}}{2} \tag{2.1}$$

and

$$A > \frac{\ln \lambda_1}{\lambda_1} - \frac{1 + \sqrt{1 + 2\theta - 2\theta\lambda_1 M}}{\theta\lambda_1} \tag{2.2}$$

respectively, where

$$\Theta = \frac{e^{\lambda_1\theta\alpha} - \lambda_1\theta\alpha - 1}{(\lambda_1\theta)^2}$$

and

$$M = \frac{1-\alpha-\sqrt{(1-\alpha)^2-4\Theta}}{2}.$$

Remark 2.1. ([17], [35]) Observe that when $\theta = 1$, then $\Theta = \frac{\lambda_1-\lambda_1\alpha-1}{\lambda_1{}^2}$, and (2.1) reduces to

$$A > 2\alpha + \frac{2}{\lambda_1} - 1, \qquad (C_{12})$$

while in this case it follows that $M = 1-\alpha-\frac{1}{\lambda_1}$ and (2.2) reduces to

$$A > \frac{\ln \lambda_1 - 1 + \sqrt{5-2\lambda_1+2\alpha\lambda_1}}{\lambda_1}. \qquad (C_{13})$$

In the case where $\alpha = \frac{1}{e}$, then $\lambda_1 = e$, and (C_{13}) leads to

$$A > \frac{\sqrt{7-2e}}{e} \approx 0.459987065.$$

It is to be noted that as $\alpha \longrightarrow 0$, then all the previous conditions $(C_4) - (C_{12})$ reduce to the condition (C_2), i.e.

$$A > 1.$$

However, the condition (C_{13}) leads to

$$A > \sqrt{3} - 1 \approx 0.732,$$

which is an essential improvement. Moreover (C_{13}) improves all the above conditions when $0 < \alpha \leq \frac{1}{e}$ as well. Note that the value of the lower bound on A can not be less than

$$\frac{1}{e} \approx 0.367879441.$$

Thus the aim is to establish a condition which leads to a value *as close as possible to* $\frac{1}{e}$. For illustrative purpose, we give the values of the lower bound on A under these conditions when $\alpha = \frac{1}{e}$.

(C_4):	0.966166179
(C_5):	0.892951367
(C_6):	0.863457014
(C_7):	0.845181878
(C_8):	0.735758882
(C_9):	0.709011646
(C_{10}):	0.708638892
(C_{11}):	0.599215896
(C_{12}):	0.471517764
(C_{13}):	0.459987065

We see that the condition (C_{13}) essentially improves all the known results in the literature.

Example 2.1. ([35]) Consider the delay differential equation

$$x'(t) + px(t - q\sin^2\sqrt{t} - \frac{1}{pe}) = 0,$$

where $p > 0$, $q > 0$ and $pq = 0.46 - \frac{1}{e}$. Then

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} pds = \liminf_{t\to\infty} p(q\sin^2\sqrt{t} + \frac{1}{pe}) = \frac{1}{e}$$

and

$$A = \limsup_{t\to\infty} \int_{\tau(t)}^{t} pds = \limsup_{t\to\infty} p(q\sin^2\sqrt{t} + \frac{1}{pe}) = pq + \frac{1}{e} = 0.46.$$

Thus, according to Remark 2.1, all solutions of this equation oscillate. Observe that none of the conditions (C_4)-(C_{12}) apply to this equation.

Following this historical (and chronological) review we also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}$$

this problem has been studied by Domshlak [2], Elbert and Stavroulakis [7], Kozakiewicz [20], Li [31,32], Domshlak and Stavroulakis [5], Tang and Yu [40], Yu and Tang [46] and Tang, Yu and Wang [41].

In 1986, Domshlak [2] first observed the following special critical situation: *Among the equations of the form*

$$x'(t) + p(t)x(t-\tau) = 0, \quad t \geq t_0, \tag{1)'$$

with

$$\lim_{t\to\infty} p(t) = \frac{1}{\tau e}$$

there exist equations such that their solutions are oscillatory in spite of the fact that the corresponding "limiting" equation

$$x'(t) + \frac{1}{\tau e}x(t-\tau) = 0, \quad t \geq t_0$$

admits a non-oscillatory solution, namely $x(t) = e^{-t/\tau}$.

In 1996, Domshlak and Stavroulakis [5] obtained the following results in the special critical case $\liminf_{t\to\infty} p(t) = 1/\tau e$.

Theorem 2.1. ([5]) (i) Assume that

$$\liminf_{t\to\infty} p(t) = \frac{1}{\tau e}, \quad \liminf_{t\to\infty} \left[\left(p(t) - \frac{1}{\tau e}\right)t^2\right] = \frac{\tau}{8e},$$

and

$$\liminf_{t\to\infty} \left\{\left[\left(p(t) - \frac{1}{\tau e}\right)t^2 - \frac{\tau}{8e}\right]\ln^2 t\right\} > \frac{\tau}{8e}.$$

Then all solutions of Eq. (1)′ oscillate.

(ii) Assume that for sufficiently t

$$p(t) \leq \frac{1}{\tau e} + \frac{\tau}{8et^2}\left(1 + \frac{1}{\ln^2 t}\right).$$

Then Eq. (1)′ has an eventually positive solution.

In 1998 Diblik [1] generalized this theorem as follows:

Set $\ln_1 t = \ln t$, $\ln_{k+1} t = \ln(\ln_k t)$, $\quad k = 1, 2, \cdots$

Theorem 2.2. ([1]) (i) Assume that for an integer $k \geq 2$ and a constant $\theta > 1$

$$p(t) \geq \frac{1}{\tau e} + \frac{\tau}{8et^2}\Big[1 + (\ln_1 t)^{-2} + (\ln_1 t \ln_2 t)^{-2} + \cdots$$

$$+(\ln_1 t \ln_2 t \cdots \ln_{m-1} t)^{-2} + \theta(\ln_1 t \ln_2 t \cdots \ln_m t)^{-2} \Big], \text{ as } t \to \infty.$$

Then all solutions of Eq. $(1)'$ oscillate.

(ii) Assume that for a positive integer k

$$p(t) \le \frac{1}{\tau e} + \frac{1}{8et^2}\left[1 + (\ln_1 t)^{-2} + (\ln_1 t \ln_2 t)^{-2} + \cdots + (\ln_1 t \ln_2 t \cdots \ln_m t)^{-2}\right],$$

as $t \to \infty$.

Then there exists a positive solution $x = x(t)$ of Eq. $(1)'$. Moreover

$$x(t) < e^{\frac{-t}{\tau}} \sqrt{t \ln t \ln_2 t \ldots \ln_k t}, \text{ as } t \to \infty.$$

Definition 2.1. ([7]) The piecewise continuous function $p : [t_0, \infty) \to [0, \infty)$ belongs to $\mathcal{A}_\lambda$ if

$$\int_{\tau(t)}^{t} p(s)ds \ge \frac{1}{e} \quad \text{for sufficiently large } t$$

and

$$\int_{\tau(t)}^{t} p(s)ds - \frac{1}{e} \ge \lambda_k \left(\int_{t_k}^{t_{k+1}} p(s)ds - \frac{1}{e} \right) \quad \text{for} \quad t_k < t \le t_{k+1}, \quad k = 1, 2, \cdots,$$

for some $\lambda_k \ge 0$, and $\liminf_{k \to \infty} \lambda_k = \lambda > 0$.

In 1995, Elbert and Stavroulakis [7] proved the following theorem.

Theorem 2.3. ([7]) Assume that $\tau(t)$ is strictly increasing on $[t_0, \infty)$ and that $p(t) \in \mathcal{A}_\lambda$ for some $\lambda \in (0, 1]$ and either

$$\lambda \limsup_{k \to \infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e} \right) > \frac{2}{e} \tag{2.3}$$

or

$$\lambda \liminf_{k \to \infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e} \right) > \frac{1}{2e}. \tag{2.4}$$

Then all solutions of Eq.(1) oscillate.

In [7], Elbert and Stavroulakis put forth the following open problem.

Open Problem 2.1. Whether or not the upper bounds in the conditions (2.3) and (2.4) of Theorem 2.3 can be replaced by smaller ones.

In 2000, Tang and Yu [40] and in 2001, Yu and Tang [46] gave an answer to this open problem by improving the above conditions (2.3) and (2.4) as follows:

$$\lambda \limsup_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e} \right) > \frac{1}{e} \tag{2.3$'$}$$

and

$$\lambda \liminf_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e} \right) > \frac{1}{8e}. \tag{2.4$'$}$$

REFERENCES

1. J. Diblik, Positive and oscillating solutions of differential equations with delay in critical case, *J. Comput. Appl. Math.* **88** (1998), 185-2002.

2. Y. Domshlak, Sturmian comparison method in oscillation study for discrete difference equations, I, *J. Diff. Integr. Eqs*, **7** (1994), 571-582.

3. Y. Domshlak, Delay-difference equations with periodic coefficients: sharp results in oscillation theory, *Math. Inequal. Appl.*, **1** (1998), 403-422.

4. Y. Domshlak and A. Aliev, On oscillatory properties of the first order differential equations with one or two retarded arguments, *Hiroshima Math. J.* **18** (1998), 31-46.

5. Y. Domshlak and I.P. Stavroulakis, Oscillations of first-order delay differential equations in a critical state, *Applicable Anal.*, **61** (1996), 359-377.

6. A. Elbert and I.P. Stavroulakis, Oscillations of first order differential equations with deviating arguments, Univ of Ioannina T.R. No 172 1990, *Recent trends in differential equations*, 163-178, World Sci. Ser. Appl. Anal.,1, World Sci. Publishing Co. (1992).

7. A. Elbert and I. P. Stavroulakis, Oscillation and non-oscillation criteria for delay differential equations, *Proc. Amer. Math. Soc.*, **123** (1995), 1503-1510.

8. L.H. Erbe, Qingkai Kong and B.G. Zhang, Oscillation Theory for Functional Differential Equations, Marcel Dekker, New York, 1995.

9. L.H. Erbe and B.G. Zhang, Oscillation of first order linear differential equations with deviating arguments, *Differential Integral Equations*, **1** (1988), 305-314.

10. N. Fukagai and T. Kusano, Oscillation theory of first order functional differential equations with deviating arguments, *Ann. Mat. Pura Appl.*,**136** (1984), 95-117.

11. K.Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic Publishers, 1992.

12. I. Gyori and G. Ladas. Oscillation Theory of Delay Differential Equatiosn with Applications, Clarendon Press, Oxford, 1991.

13. J.K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1997.

14. A.F. Ivanov, and V.N. Shevelo, Oscillation and asymptotic behavior of solutions of first order differential equations, *Ukrain. Math. Zh.*, **33** (1981), 745-751, 859.

15. J. Jaroš and I.P. Stavroulakis, Oscillation tests for delay equations, *Rocky Mountain J. Math.*, **29** (1999), 139-145.

16. C. Jian, Oscillation of linear differential equations with deviating argument, *Math. in Practice and Theory*, **1** (1991), 32-41 (in Chinese).

17. M. Kon, Y.G. Sficas and I.P. Stavroulakis, Oscillation criteria for delay equations, *Proc. Amer. Math. Soc.*, **128** (2000), 2989-2997.

18. R.G. Koplatadze and T.A. Chanturija, On the oscillatory and monotonic solutions of first order differential equations with deviating arguments, *Differentsial'nye Uravneniya*, **18** (1982), 1463-1465.

19. R.G. Koplatadze and G. Kvinikadze, On the oscillation of solutions of first order delay differential inequalities and equations, *Georgian Math. J.* **1** (1994), 675-685.

20. E. Kozakiewicz, Conditions for the absence of positive solutions of first order differential inequalities with deviating agruments, 4th Intl. Coll. on Differential Equations, VSP, 1994, 157-161.

21. M.R. Kulenovic and M.K. Grammatikopoulos, First order functional differential inequalities with oscillating coefficients, *Nonlinear Anal.* **8** (1984), 1043-1054.

22. M.K. Kwong, Oscillation of first order delay equations, *J. Math. Anal. Appl.*, **156** (1991), 374-286.

23. G. Ladas, Sharp conditions for oscillations caused by delay, *Applicable Anal.*, **9** (1979), 93-98.

24. G. Ladas, V. Laskhmikantham and J.S. Papadakis, Oscillations of higher-order retarded differential equations generated by retarded arguments, *Delay and Functional Differential Equations and Their Applications.* Academic Press, New York, 1972, 219-231.

25. G. Ladas, C. Qian and J. Yan, A comparison result for the oscillation of delay differential equations, *Proc. Amer. Math. Soc.*, **114** (1992), 939-946.

26. G. Ladas, Y.G. Sficas and I.P. Stavroulakis, Functional-differential inequalities and equations with oscillating coefficients, *Trends in theory and practice of nonlinear differential equations,* (Arlington, Tex., 1982), 277–284, Lecture Notes in Pure and Appl. Math., 90, Marcel Dekker, New York, 1984.

27. G.S. Ladde, Oscillations caused by retarded perturbations of first order linear ordinary differential equations, *Atti Acad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur.*, **63** (1977), 351-359.

28. G.S. Ladde, Class of functional equations with applications, *Nonlinear Anal.*, **2** (1978), 259-261.

29. G.S. Ladde, Stability and oscillation in single-species processes with past memory, *Int. J. System Sci.*, **10** (1979), 621-647.

30. G.S. Ladde, V. Lakshmikantham and B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, Marcel Dekker, New York, 1987.

31. B. Li, Oscillations of delay differential equations with variable coefficients, *J. Math. Anal. Appl.*, **192** (1995), 312-321.

32. B. Li, Oscillations of first order delay differential equations, *Proc. Amer. Math. Soc.*, **124** (1996), 3729-3737.

33. A.D. Myshkis, Linear homogeneous differential equations of first order with deviating arguments, *Uspekhi Mat. Nauk*, **5** (1950), 160-162 (Russian).

34. Ch.G. Philos and Y.G. Sficas, An oscillation criterion for first-order linear delay differential equations, *Canad. Math. Bull.* **41** (1998), 207-213.

35. Y.G. Sficas and I.P. Stavroulakis, Oscillation criteria for first-order delay equations, *Bull. London Math. Soc.*, **35** (2003), 239-246.

36. V.A.Staikos and I.P. Stavroulakis, Bounded oscillations under the effect of retardations for differential equations of arbitrary order, *Proc. Roy. Soc. Edinburgh Sect A*, **77** (1977), no.1-2, 129-136.

37. X.H. Tang, Oscillation of delay differential equations with variable coefficients, *J. Math. Study,* **31** (3)(1998), 290-293.

38. X.H. Tang and J.S. Yu, Oscillations of first order delay differential equations in a critical state, *Mathematica Applicata* **13** (1)(2000), 75-79.

39. X.H. Tang and J.S. Yu, Oscillation of first order delay differential equations with oscillating coefficients, *Appl Math.*-JCU, **15** (2000), 252-258.

40. X.H. Tang and J.S. Yu, Oscillation of first order delay differential equations, *J. Math. Anal. Appl.* **248** (2000), 247-259.

41. X.H. Tang, J.S. Yu and Z.C. Wang, Comparison theorems of oscillation of first order delay differential equations in a critical state with applications, *Ke Xue Tongbao*, **44** (1999), 26-30.

42. A. Tomaras, Oscillation behavior of an equation arising from an industrial probrem, *Bull. Austral. Math. Soc.*, **13** (1975), 255-260.

43. A. Tomaras, Oscillations of a first order functional differential equation, *Bull. Austral. Math. Soc.*, **17** (1977), 91-95.

44. A. Tomaras, Oscillatory behaviour of first order delay differential equations, *Bull. Austral. Math. Soc.*,**19** (1978), 183-190.

45. Z.C. Wang, I.P. Stavroulakis and X.Z. Qian, A survey on the oscillation of solutions of first order linear differential equations with deviating arguments, *Appl. Math. E-Notes*, **2** (2002), 171-191.

46. J.S. Yu and X.H. Tang, Comparison theorems in delay differential equations in a critical state and application, *Proc. London Math. Soc.*, **63** (2001),188-204.

47. J.S. Yu and Z.C. Wang, Some further results on oscillation of neutral differential equations, *Bull. Austral. Math. Soc.*, **46** (1992), 149-157.

48. J.S. Yu, Z.C. Wang, B.G. Zhang and X.Z. Qian, Oscillations of differential equations with deviting arguments, *Panamerican Math. J.*, **2** (1992), 59-78.

49. Y. Zhou and Y.H.Yu, On the oscillation of solutions of first order differential equations with deviating arguments, *Acta Math. Appl. Sinica* **15**, no.3, (1999), 288-302.